arXiv:hep-th/9506013v1 2 Jun 1995

OU-HET 198

May 31, 1995

Effective Topological Theory for Gravitational Anyon Scatterings at Ultra-High Energies

Takahiro KUBOTA and Hiroyuki TAKASHINO

Department of Physics, Osaka University,

Toyonaka, Osaka 560, Japan

Abstract

The idea of the effective topological theory for high-energy scattering proposed by H. and E. Verlinde is applied to the $(2+1)$ dimensional gravity with Einstein action plus Chern-Simons terms. The calculational steps in the topological description are compared with the eikonal approximation. It is shown that the Lagrangian of the effective topological theory turns out to vanish except for boundary terms.

§1. Introduction

It has sometimes happened in physics that very peculiar phenomena show up under extreme conditions, *e.g.*, asymptotic freedom in high energy QCD scatterings, superconductivity at low temperature, strong gravitational forces near black holes and so forth. There are therefore good reasons to feel free to do "Gedanken Experiment" to uncover deep structures of theories at our hands. As one of such Gedanken Experiments, there have been several interesting attempts [1)−4)] of inspecting Planckian energy scatterings. This is expected to serve as a tool of inspecting quantum aspects of gravitational theories, in particular, in the framework of string theories.

There have been principally three methods to study the Planckian energy scattering, namely, (i) the eikonal approximation [1),3)], (ii) the shock wave method [2),5)] and (iii) effective topological theory proposed by H. and E. Verlinde [6)]. The field-theoretical formulation of the eikonal approximation has been known [7),8)] for long time, while the shock wave method has come to our concern rather recently. It has been known that high-energy small-angle scattering amplitudes computed in the eikonal and shock wave methods always agree. The third method of Verlindes on the other hand is motivated in such a way that the most dominant terms contributing to the leading eikonal approximation are separated from the outset on the Lagrangian level. A very peculiar fact in four dimensional Einstein gravity at the Planckian energy is that the obtained effective Lagrangian turns out to be topological [9)], *i.e.*, being expressed as a BRST-exact form up to surface integrals. Kabat and Ortiz [10)] examined the effective topological method, comparing it with the other methods in four dimensional Einstein gravity.

Now that we have various methods of producing the same leading approximation, an imminent issue is how to go beyond the leading approximation by including non-leading terms. This is important because several interesting quantum effects could

emerge there. There have been a lot of works in this direction in the first method [11),12)]. As far as Verlindes' topological method is concerned, however, it does not look straightfoward to improve their approximation method [11)].

In the present paper, we do not try to go beyond the leading approximation in Verlindes' method, but rather we would like to reexamine the calculational steps of their method by paying particular attentions to the connection between Verlindes' and the conventional eikonal methods. We will apply the Verlindes' method to the (2+1)-dimensional Einstein action supplemented by Chern-Simons terms. Although this gravitational theory is a theoretical laboratory, it is interesting in its own right since it produces fractional spin and statistics [13)] as in the vector Chern-Simons theories [14),15)]. We will show that the (2+1)-dimensional Einstein-Chern-Simons gravity is in fact described effectively at high energies by a topological field theory, whose Lagrangian turns out to vanish except for boundary terms.

§2. The Gravitational Anyon

As a starting point, let us begin with the total action

$$S = S_E + S_{CS} + S_{matter}, \tag{1}$$

where the Einstein and Chern-Simons actions are given respectively by

$$S_E = \frac{1}{2\kappa^2} \int d^3x \sqrt{g^{(3)}} R^{(3)}, \tag{2}$$

$$S_{CS} = \frac{1}{4\kappa^2 \mu} \int d^3x \epsilon^{\lambda\mu\nu} \Gamma^{\sigma}_{\lambda\rho} (\partial_\mu \Gamma^{\rho}_{\sigma\nu} + \frac{2}{3} \Gamma^{\rho}_{\mu\tau} \Gamma^{\tau}_{\nu\sigma}). \tag{3}$$

The mass dimensions of the gravitational constant κ^2 in three space-time dimensions is $\dim[\kappa^2] = -1$.

The (2+1)-dimensional gravity without the Chern-Simons term has been a subject of intensive research in the last decade [16)]. This theory is peculiar in that there

is no graviton and that space-time is flat outside sources. Dynamics is determined by global geometry. Scattering problems in this theory was also investigated in detail [17]. Inclusion of the Chern-Simons term renders the theory "topologically massive" with propagating modes [18]. The Chern-Simons action contains terms of third derivatives with respect to the metric, and one may expect that these terms would be of importance for high-energy scatterings. This is in contrast with the vector Chern-Simons action where only first-derivative terms are contained.

For the sake of simplicity, we consider the matter action consisting of only a scalar field interacting with gravitational field.

$$\begin{aligned} S_{matter} &= \int d^3x \sqrt{g^{(3)}} (g^{(3)\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi) \\ &= \int d^3x [\eta^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi \\ &\quad + h^{\mu\nu} \{ -\partial_\mu \phi^\dagger \partial_\nu \phi + \frac{1}{2} \eta_{\mu\nu} (\partial^\lambda \phi^\dagger \partial_\lambda \phi - m^2 \phi^\dagger \phi) \}] + \mathcal{O}(h^2). \end{aligned} \tag{4}$$

Here we have expanded the metric around the flat space-time by setting

$$g^{(3)}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1). \tag{5}$$

The higher-order terms with respect to fluctuations $h_{\mu\nu}$ are not considered in the leading eikonal approximation.

The Chern-Simons interaction (3) is known to produce fractional statistics [13], which is in complete analogy with the case of gauge interactions [14,15]. The emergence of the fractional statistics is ascribed to the fact that a non-trivial phase shows up when two point-particles are interchanged adiabatically. A simple way to evaluate the phase is to look at the two-body static potential, or more generally the two-point function of $h_{\mu\nu}$

$$\Delta_{\mu\nu,\lambda\rho}(x, y) = \frac{-i}{8\kappa^2} < h_{\mu\nu}(x) h_{\lambda\rho}(y) > . \tag{6}$$

Now as a preliminary calculation for later use, we present here the propagator

when we take the gauge fixing condition specified by

$$\mathcal{L}_{g.f.} = -\frac{1}{4\kappa^2\xi}(\partial_\lambda h^{\lambda\mu} - \frac{1}{2}\partial^\mu h^\lambda_\lambda)^2. \tag{7}$$

(The propagator in the Landau gauge has been given in Ref. 18).) The defining equation of the two-point function becomes

$$\begin{aligned}
[A^{\mu\nu\lambda\rho}\partial^2 &+ \frac{1}{2\mu}B^{\mu\nu\lambda\rho} - (1-\frac{1}{\xi})C^{\mu\nu\lambda\rho} + 2(1-\frac{1}{\xi})D^{\mu\nu\lambda\rho} \\
&- (1-\frac{1}{\xi})\eta^{\mu\nu}\eta^{\lambda\rho}\partial^2]\Delta_{\lambda\rho,\sigma\tau}(x,y) = \frac{1}{2}(\eta^\mu_\sigma\eta^\nu_\tau + \eta^\mu_\tau\eta^\nu_\sigma)\delta^3(x-y),
\end{aligned} \tag{8}$$

where we have introduced the following notations

$$A^{\mu\nu\lambda\rho} = \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\rho}, \tag{9}$$

$$B^{\mu\nu\lambda\rho} = \epsilon^{\nu\sigma\rho}(\eta^{\mu\lambda}\partial^2 - \partial^\mu\partial^\lambda)\partial_\sigma + \epsilon^{\mu\sigma\rho}(\eta^{\nu\lambda}\partial^2 - \partial^\nu\partial^\lambda)\partial_\sigma + (\lambda \longleftrightarrow \rho), \tag{10}$$

$$C^{\mu\nu\lambda\rho} = \eta^{\nu\rho}\partial^\mu\partial^\lambda + \eta^{\mu\rho}\partial^\nu\partial^\lambda + (\lambda \longleftrightarrow \rho), \tag{11}$$

$$D^{\mu\nu\lambda\rho} = \eta^{\lambda\rho}\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial^\lambda\partial^\rho. \tag{12}$$

The formal solution to (8) is given by

$$\begin{aligned}
\Delta_{\lambda\rho,\sigma\tau}(x,y) = &[\frac{1}{4}A_{\lambda\rho\sigma\tau}(\frac{1}{\partial^2} - \frac{1}{\partial^2+\mu^2}) - \frac{\mu}{8}B_{\lambda\rho\sigma\tau}\frac{1}{(\partial^2+\mu^2)(\partial^2)^2} \\
&+\frac{1}{4}C_{\lambda\rho\sigma\tau}\{\frac{1}{(\partial^2+\mu^2)\partial^2} + (\xi-1)\frac{1}{(\partial^2)^2}\} \\
&-\frac{1}{4}D_{\lambda\rho\sigma\tau}\frac{1}{(\partial^2+\mu^2)\partial^2} - \frac{1}{4}\frac{1}{(\partial^2+\mu^2)(\partial^2)^2}\partial_\lambda\partial_\rho\partial_\sigma\partial_\tau \\
&-\frac{1}{4}\eta_{\lambda\rho}\eta_{\sigma\tau}\frac{1}{\partial^2}]\delta^3(x-y).
\end{aligned} \tag{13}$$

The anyonic interaction comes from the second term in (13). In fact for the static case, it produces an interaction

$$\begin{aligned}
&-\frac{\mu}{8}B_{000i}\frac{1}{(-\nabla^2+\mu^2)(-\nabla^2)^2}\delta^2(\mathbf{x}-\mathbf{y})\delta(x^0-y^0) \\
&= \frac{1}{8\pi}\epsilon_{0ij}\frac{x^j-y^j}{|\mathbf{x}-\mathbf{y}|}\{K_0'(\mu|\mathbf{x}-\mathbf{y}|) + \frac{1}{\mu|\mathbf{x}-\mathbf{y}|}\}\delta(x^0-y^0).
\end{aligned} \tag{14}$$

Here K_0 is the modified Bessel function. The two-body potential is obtained by multiplying (14) by the relative velocity $\frac{d}{dt}(x^i - y^i)$. As $|\mathbf{x} - \mathbf{y}| \to \infty$, the modified Bessel function decreases exponentially, and the second term in the brackets in (14) is more dominant. This term induces a phase proportional to $\kappa^2 m^2/\mu$ as the two particles are interchanged.

§3. The Eikonal Approximation

In the following we summarize the eikonal approximation method in the gravitational theory (1). Some of the contents in this section was implicitly stated in Ref. 19), but will be given below for the sake of comparison with the topological field theory. The technique of eikonal approximation has been well-known and explained in quantum mechanics textbooks. The analogous calculations in quantum field theories have also been studied in the sixties. Roughly speaking, the field theoretical eikonal approximation amounts to summing up an infinite number of Feynman diagrams of exchange type (Fig. 1) without worring about self-energies or vertex renormalzation effects, either.

Hereafter we will use the formulation proposed by Abarbanel and Itzykson [7]. The linearized gravitational interaction in Eq. (4) motivates us to consider an operator

$$A(X,P) = P_\mu\{h^{\mu\nu}(X) - \frac{1}{2}\eta^{\mu\nu}h^\lambda_\lambda(X)\}P_\nu + \frac{1}{2}m^2 h^\lambda_\lambda(X), \tag{15}$$

where X_μ and P_ν are assumed to satisfy formally the usual commutation relation $[X_\mu, P_\nu] = -i\eta_{\mu\nu}$.

Each of the two particles in Fig. 1 is propagating while emitting and/or absorbing virtual gravitons due to the linearized interactions. The evaluation of the propagator is facilitated by making use of the formula

$$\lim_{P^2\to m^2}(P^2-m^2)\frac{1}{P^2-m^2-A(X,P)+i\epsilon}(P^2-m^2)$$
$$= T\exp\{-i\int_0^\infty d\tau A(X+2\tau P,P)\}A(X,P). \tag{16}$$

Here T is the ordering according to the variable τ. The matter propagator in Fig. 1 may be obtained just by sandwitching the above formula by initial and final states.

The Feynman diagrams in Fig. 1 are evaluated by connecting the gravitational lines by the propagator (6). The combinatorial factors are such that the invariant amplitude is expressed by the following exponentiated formula

$$\begin{aligned}
&\frac{-i}{(2\pi)^3}\mathcal{T}(s,t)\delta^3(p_2+p_2'-p_1-p_1') \\
= \ &\exp\{8\kappa^2 i\int\int d^3y d^3y'\frac{\delta}{\delta h_{\mu\nu}(y)}\Delta_{\mu\nu,\lambda\rho}(y,y')\frac{\delta}{\delta h'_{\lambda\rho}(y')}\} \\
&\times < p_2|T\exp\{-i\int_0^\infty d\tau A(X+2\tau P,P)\}A(X,P)|p_1 > \\
&\times < p_2'|T\exp\{-i\int_0^\infty d\tau' A'(X+2\tau' P,P)\}A'(X,P)|p_1' > |_{h=h'=0}.
\end{aligned} \tag{17}$$

Here s and t are the usual Mandelstam variables.

To get the small angle scattering amplitude, all we have to do is to replace the operator P by the average of the initial and final momenta *i.e.*, $p=\frac{1}{2}(p_1+p_2)$, $p'=\frac{1}{2}(p_1'+p_2')$. After this manipulation we arrive at the eikonal amplitude (to be denoted hereafter by $\mathcal{T}_E$),

$$\begin{aligned}
\mathcal{T}_E(s,t) \ = \ &-2i\bar{s}\int db\exp\{ib\cdot(p_2-p_1)\} \\
&\times[\exp\{-8\kappa^2 i\int_{-\infty}^\infty d\tau\int_{-\infty}^\infty d\tau'(p^\mu p^\nu-\frac{1}{2}\eta^{\mu\nu}(p^2-m^2)) \\
&\times\Delta_{\mu\nu,\lambda\rho}(b+2p\tau-2p'\tau',0)(p'^\lambda p'^\rho-\frac{1}{2}\eta^{\lambda\rho}(p'^2-m^2))\} \\
&-1].
\end{aligned} \tag{18}$$

Here b is a vector satisfying the orthogonality $b \cdot p = b \cdot p' = 0$ and we have also introduced a notation

$$\bar{s} = s\sqrt{1 - \frac{4m^2 - t}{s}}. \tag{19}$$

Keeping terms up to linear in t/s and m^2/s, we get

$$\mathcal{T}_E(s,t) \sim -2i\bar{s}\int_{-\infty}^{+\infty} db \exp(\pm i\sqrt{-t}b)[\exp\{\frac{i}{2}\kappa^2\bar{s}\Delta(b)\} - 1], \tag{20}$$

where

$$\Delta(b) = \frac{1}{2}|b| + \frac{1}{2\mu}\epsilon(b) + \frac{1}{2\mu}\exp\{-\mu|b|\} - \frac{1}{2\mu}\epsilon(b)\exp\{-\mu|b|\}. \tag{21}$$

Here $\pm$ in the exponent in (20) indicates that, if the deflection angle after scattering is positive (negative) in the center of mass system, then we should take minus (plus) sign. Note that the gauge parameter dependence disappeared in this approximation. We can see easily that the inequality $\Delta(b) \neq \Delta(-b)$ produces aymmetry of the amplitudes with respect to the deflection angle. The asymmetric phase in (20) is analogous to the Aharonov-Bohm effect discussed in sec. 2. There is, however, slight difference in that, while the phase of the fractional statistics is proportional to $\kappa^2 m^2/\mu$, the counterpart in (20) is proportional to $\kappa^2\bar{s}/\mu$. Incidentally, note that $\Delta(b)$ satisfies the differential equation

$$(\frac{d^2}{db^2} - \frac{1}{\mu}\frac{d^3}{db^3})\Delta(b) = \delta(b). \tag{22}$$

§4. A Qualitative Analysis towards Topological Description

Here we digress a little while to discuss the eikonal formula (20), and its implications. In passing from (18) to (20), we have contracted Lorentz indices term by term. The calculation is straightforward but rather tedious. There is, however, more direct way to reach (20).

Let us make use of the light-cone variables, i.e., $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$, $x^{\perp} = x^2$. We will call the $\pm$-direction longitudinal, while $\perp$-direction transverse. Suppose that the momenta p^{μ} and p'^{μ} lie in the longitudinal direction at the ultra-high energy. The most dominant component of p^{μ} (p'^{μ}) is p^+ (p'^-). The summation over the Lorentz indices is then very much simplified, and the only contribution is reduced to the longitudinal component $\Delta_{++,--}(b + 2p\tau - 2p'\tau', 0)$.

The dynamics under the eikonal approximation is that the particle interactions occur at short distance in the longitudinal direction, i.e., $2p\tau - 2p'\tau' \sim 0$, which is in contrast with the rather large distance interaction in the transverse component. Considering these facts we may put the formal solution of the Green's function (13) in the following manner. The derivatives in the transverse direction may be important and must be kept throughout. Those in the longitudinal direction, on the other hand, pick up small corrections in the ultra-high energy scatterings and may be discarded in our problem. These approximation amounts to the following

$$\begin{aligned}
\Delta_{++,--}(x,y) &\sim [\frac{1}{4}A_{++--}(\frac{1}{-\partial_{\perp}^2} - \frac{1}{-\partial_{\perp}^2 + \mu^2}) \\
&\quad -\frac{\mu}{8}B_{++--}\frac{1}{(-\partial_{\perp}^2 + \mu^2)(-\partial_{\perp}^2)^2}]\delta^3(x-y) \\
&= -\frac{1}{2}(\partial_{\perp}^2 - \frac{1}{\mu}\partial_{\perp}^3)^{-1}\delta^3(x-y).
\end{aligned} \tag{23}$$

Here we have put $A_{++--} = 2$ and $B_{++--} \sim 4\partial_{\perp}^3$. We immediately notice that the same differential operator has emerged in (23) as in (22). In other words, the Green function is replaced by

$$\Delta_{++,--}(b + 2p\tau - 2p'\tau', 0) \sim -\frac{1}{2}\Delta(b)\delta^2(2p\tau - 2p'\tau') \tag{24}$$

and thereby we can easily jump from (18) to (20).

The separation of the dynamics into longitudinal and transverse sectors suggests a new way of looking at the eikonal approximation. It is all in the "transverse" Green function $\Delta(b)$ where the dynamics of the scatterings is contained. Apparently it is

possible to extract $\Delta(b)$ on the Lagrangian level by sorting out the most dominant ones among the kinetic terms. It is, however, not quite obvious whether the effective Lagrangian after sorting out is a topological field theory. This is what we would like to see next and will show that the effective Lagrangian is non-vanishing only on boundaries.

§5. Effective Topological Field Theories

Now we are in a position to discuss the same scattering problem in the effective topological method proposed by Verlindes [6]. We would like to see to what extent the Verlindes' idea works for the action (1). As we have seen, the factorization of the dynamics along the longitudinal and transverse direction simplifies the problem considerably, and we are led to take the following gauge choice

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$g_{\alpha\beta}$}} & 0 \\ & & 0 \\ 0 & 0 & h \end{pmatrix}, \qquad (\alpha, \beta = 0, 1). \tag{25}$$

It is assumed that h is space-time independent and is just a constant. Physical quantities should not depend on h. The ghost action in the case of the above gauge choice becomes

$$S_{gh} = \int d^3x \sqrt{g^{(3)}} \{ b_{2\alpha} (\nabla^2 c^\alpha + \nabla^\alpha c^2) + 2 b_{22} \nabla^2 c^2 \}. \tag{26}$$

Previously, in order to reach Eq. (20) we have neglected the derivatives along the longitudinal direction and have kept only those in the transverse one in the propagators. This procedure may be achieved equivalently on the Lagrangian level by considering scaling behaviors under the change of the metric

$$g_{\alpha\beta} \to l_\parallel^2 g_{\alpha\beta}, \qquad h \to l_\perp^2 h. \tag{27}$$

Here $l_{\|}$ and $l_{\perp}$ are the typical length scales characterizing the longitudinal and transversal dynamics, respectively. In order to see the meaning of the scaling properties under (27) , let us separate the Einstein, Chern-Simons and ghost actions into two parts, *i.e.,* longitudinal and transverse ones,

$$S_E = S_{E\|} + S_{E\perp}, \quad S_{CS} = S_{CS\|} + S_{CS\perp}, \quad S_{gh} = S_{gh\|} + S_{gh\perp}. \tag{28}$$

Each term in (28) is defined by

$$\begin{aligned}
S_{E\|} &= \frac{1}{8\kappa^2}\int d^3x\sqrt{-g}\sqrt{-h}h^{-1}(\partial_2 g_{\alpha\beta})(\partial_2 g_{\gamma\delta})(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}), && (29)\\
S_{E\perp} &= \frac{1}{2\kappa^2}\int d^3x\sqrt{-g}\sqrt{-h}R_g, && (30)\\
S_{CS\|} &= \frac{1}{4\kappa^2\mu}\int d^3x\epsilon^{\alpha\beta 2}\{\Gamma^{\gamma}_{\beta 2}(\partial_2\Gamma^2_{\gamma\alpha}) - \Gamma^2_{\alpha\gamma}(\partial_2\Gamma^{\gamma}_{\beta 2})\}, && (31)\\
S_{CS\perp} &= \frac{1}{4\kappa^2\mu}\int d^3x\epsilon^{\alpha\beta 2}\{\Gamma^{\gamma}_{\delta\beta}(\partial_2\Gamma^{\delta}_{\gamma\alpha}) + \partial_\beta(\Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\gamma 2})\}, && (32)\\
S_{gh\|} &= \int d^3x\sqrt{-g}\sqrt{-h}h^{-1}\{b_{2\alpha}(\partial_2 c^\alpha) + 2b_{22}(\partial_2 c^2)\}, && (33)\\
S_{gh\perp} &= -\int d^3x\sqrt{-g}\sqrt{-h}g^{\alpha\beta}(\nabla_\alpha b_{2\beta})c^2. && (34)
\end{aligned}$$

Here R_g in (30) denotes the two-dimensional scalar curvature associated with $g_{\alpha\beta}$.

As the notations show, the longitudinal and transverse parts of the actions are tansformed distinctively under (27) *i.e.,*

$$S_{E\|} \to \frac{l_{\|}^2}{l_{\perp}} S_{E\|}, \qquad S_{E\perp} \to l_{\perp} S_{E\perp}, \tag{35}$$

$$S_{CS\|} \to (\frac{l_{\|}}{l_{\perp}})^2 S_{CS\|}, \qquad S_{CS\perp} \to S_{CS\perp}, \tag{36}$$

$$S_{gh\|} \to \frac{l_{\|}^2}{l_{\perp}} S_{gh\|}, \qquad S_{gh\perp} \to l_{\perp} S_{gh\perp}. \tag{37}$$

The scaling behavior (36) differs from those in (35) and (37). We should, however, recall the fact that S_{CS} contains the mass parameter μ in front. If μ is of the same order as $1/l_{\perp}$, then the behavior (36) may be regarded as the same as (35) and (37).

In the high energy limit, the scattering dynamics is confined in the short distance region in the longitudinal direction. This means that the path integral region over the metric corresponding to $l_{\|} \ll l_{\perp}$ is most important, and $S_{E\perp}$, $S_{CS\perp}$, and $S_{gh\perp}$ may be treated classically. We will look for the local minimum of $S_{E\perp}$, $S_{CS\perp}$ and $S_{gh\perp}$ by varying the fields. The stability condition $\delta S_{E\perp} = 0$ provides us immediately with

$$g_{\alpha\beta} = \partial_\alpha X^a \partial_\beta X_a. \tag{38}$$

In other words, the conformal mode is not important and the metric is parametrized only by the two modes X^a, $(a = 1, 2)$. Another stability condition of the ghost part $\delta S_{gh\perp} = 0$ turns out to be $\nabla_\alpha b_{2\beta} = 0$ and we may set

$$b_{2\alpha} = \epsilon_{\alpha\beta 2} \partial^\beta b. \tag{39}$$

The local minimum of $S_{CS\perp}$ may be found easily by solving the equation $\partial_2 \Gamma^\beta_{\alpha\gamma} = 0$, or more simply by noting the relation

$$\delta S_{CS\perp} = \frac{1}{4\kappa^2\mu} \int d^3x \epsilon^{\alpha\beta 2} \delta g_{\beta\gamma} \nabla_\alpha \nabla^\delta (\nabla_\delta V^\gamma - \nabla^\gamma V_\delta) = 0. \tag{40}$$

Here we have introduced V^α defined by the relation

$$\partial_2 X^a + V^\alpha \partial_\alpha X^a = 0. \tag{41}$$

The most dominant configuration of X^a is realized by imposing

$$\nabla_\alpha V_\beta - \nabla_\beta V_\alpha = 0. \tag{42}$$

By putting all these conditions into the longitudinal part of the action, we arrive at

$$S_{E\|} = \frac{1}{2\kappa^2} \int d^3x \sqrt{-h} h^{-1} \epsilon_{ab} \epsilon^{\alpha\beta 2} \partial_\alpha (\partial_2 X^a \partial_\beta \partial_2 X^b), \tag{43}$$

$$S_{CS\|} = \frac{1}{2\kappa^2\mu} \int d^3x h^{-1} \epsilon^{\alpha\beta 2} \partial_\alpha (\partial_2 X^a \partial_\beta \partial_2 \partial_2 X_a). \tag{44}$$

Note that these are both in the form of total divergence.

Finally let us come to the interaction of the scalar field and gravity. The interaction part is also expressed as a surface integral

$$S_{int} = \int d^3x\sqrt{-h}\partial^\alpha(P_{a,\alpha}X^a). \tag{45}$$

Here $P_{a,\alpha} = T_{\alpha\beta}\partial^\beta X_a$ is the momentum flow defined in terms of the energy momentum tensor $T_{\alpha\beta}$. To sum up, the effective Lagrangians in (43), (44) and (45) are all expressed in the form of total divergences and hence topological.

It has been observed by Verlindes in four dimensional gravity that the Jacobian associated with the change of the path integral variables $g_{\alpha\beta} \to X^a$ is exactly cancelled by another Jacobian due to the change of the antighost. This cancellation, however, does not occur in the three dimensional case, because the Jacobian due to the change $b_{a\alpha} \to b$ is only one half of the bosonic counterpart. This incomplete cancellation of the Jacobians would have to be given due consideration if we would go beyond the leading approximation. As far as we restrict ourselves to the first approximation, however, the effect due to the Jacobian does not matter.

Another remark is on the work by Zeni [20], who applied the Verlindes' method to the Einstein gravity in three dimensions. Since Zeni did not have S_{CS} in his Lagrangian, the condition (42) did not come from (40). Instead he considered the Gauss law constraints and has obtained (42). Without imposing the constraints on the Lagrangian level, the effective theory would have been much more complicated. In our case, on the other hand, the presence of S_{CS} is crucial to get (42) and $S_{E\|}$ and $S_{CS\|}$ have therefore become the form of total divergence.

§6. Discussions

In order to see what the scattering amplitudes look like in the effective topological theory, we rewrite the effective Lagrangian in the form of contour integrals

$$
\begin{aligned}
&S_{E\|} + S_{CS\|} + S_{int} \\
&= \frac{-1}{2\kappa^2} \oint d\sigma \int \sqrt{-h} h^{-1} \frac{\partial X^a}{\partial \sigma} \{\epsilon_{ab}(\partial_2)^2 + \frac{1}{\mu\sqrt{-h}} \eta_{ab}(\partial_2)^3\} X^b \\
&+ \oint d\sigma \int \sqrt{-h} \epsilon^{\sigma\alpha 2}(P_{a,\alpha} X^a)
\end{aligned}
\tag{46}
$$

The variable σ parametrizes the boundary C of the two-dimensional manifold (x^0, x^1) as shown in Fig. 2. The variable X^a is now regarded as a function of σ and x^2 *i.e.,* $X^a = X^a(\sigma, x^2)$. The differential operator in the kinetic term of X^a is similar in form as those in Eqs. (22) and (23). Thus the combined use of the scaling argument and the semiclassical treatment of the transverse part of the action enables us to extract the Green function (21) on the Lagrangian level. It is in fact easy to see that the Green function (21) shows up in the two-point function

$$
< X^+(\sigma, b) X^-(\sigma', b') >= \frac{-i\kappa^2}{2} \Delta(-\sqrt{-h}(b - b'))\epsilon(\sigma - \sigma'). \tag{47}
$$

The path integral over X^a of the action (46) result in

$$
\begin{aligned}
&\exp\{\frac{i\kappa^2}{2} \oint d\sigma \oint d\sigma' \int db \sqrt{-h} \int db' \sqrt{-h} \epsilon^{\sigma\alpha 2} \epsilon^{\sigma'\beta 2} P_{+,\alpha}(\sigma, b) P_{-,\beta}(\sigma', b') \\
&\times \Delta(-\sqrt{-h}(b - b'))\epsilon(\sigma - \sigma')\},
\end{aligned}
\tag{48}
$$

where we have used the fact that the Green functions $< X^+ X^+ >$ and $< X^- X^- >$ both vanish. At ultra-high energies, only the components $P_{+,+}$ and $P_{-,-}$ survive the summation in the above exponent and the integrations of $P_{\pm,\pm}(\sigma, b)$ over σ and b give us the incoming and outgoing momenta of each particle. Suppose that the incident and outgoing particles have some particular relative impact parameters, then the exponentiated form in (48) is exactly the same as the integrand in (20). It is thus possible in the effective topological method to set up a calculational scheme

for scattering amplitudes. Namely scattering amplitudes are given by correlation functions of "vertex operator" $\exp\{iS_{eff}\}$ and Verlindes' program is now fulfilled for the (2+1)-dimensional Einstein-Chern-Simons gravity.

Finally a few remarks are in order with regard to previous related works in literatures. Deser, McCarthy and Steif [19)] studied the same scattering problem in the shock-wave and eikonal methods. They started with the metric of the Aichelburg-Sexl type

$$ds^2 = 2dx^+dx^- - (dx^\perp)^2 - 2F(x^-, x^\perp)(dx^-)^2. \tag{49}$$

The equation satisfied by $F(x^-, x^\perp)$ is of the same form as that of $\Delta_{++,--}$ and they examined ambiguity problems associated with the solution. The same problems also remain in the effective topological method; If one would start from the effective Lagrangian (46) and tried to get the Green function $< X^+(\sigma, b)X^-(\sigma', b') >$, the conventional $i\epsilon$-prescription is no more available and one would encounter the problem as to how to fix the boundary conditions. We are forced to go back to the original Green function (13) to fix the boundary conditions. This fact is unsatisfactory in the effective topological method.

Comparison between the topological and the conventional eikonal methods offers several implications as to the sub-leading terms. Amati, Ciafaloni and Veneziano [11)] studied sub-leading corrections to the eikonal approximation in the four dimensional gravity and have realized that the Verlindes' gauge choice has intrinsic difficulties. The problem comes from particular metric fluctuations in the off-diagonal part of the metric which are not to be gauged away. The same difficulties still remain in the (2+1)-dimensional case and we do not have much to say about them. Furthermore since some of the sub-leading terms are contained in the transverse part of the action, there seems to be little chance that the effective theory could be a topological field theory at the sub-leading order.

Acknowledgements

We would like to thank our colleagues at Osaka University for their kind interest in the present work. This work was supported in part by the Grant in Aid for Scientific Research from the Ministry of Education, Science, and Culture (Grant No. 06640396).

References

1) D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. **197B** (1987), 81; Int. J. Mod. Phys. **A3** (1988), 1615.

2) 't Hooft, Phys. Lett. **198B** (1987), 61; Nucl Phys. **B304** (1988), 867; C. Klimcik, Phys. Lett. **B 208** (1988), 373.

3) I. Muzinich and M. Soldate, Phys. Rev. **D37** (1988), 353.

4) D.J. Gross and P.F. Mende, Phys. Lett. ; **197B** (1987), 129; Nucl. Phys. **B303** (1988), 407.

5) P.C. Aichelburg and R.U. Sexl, Gen. Rel. Grav. **2** (1971), 303; T. Dray and G. 'tHooft, Nucl. Phys. **B253** (1985), 173.

6) H. Verlinde and E. Verlinde, Nucl. Phys. **B371** (1992), 246.

7) H.D.I. Abarbanel and C. Itzykson, Phys. Rev. Lett. **23** (1969), 53; M. Levy and J. Sucher, Phys. Rev. **186** (1969), 1656.

8) H. Cheng and T.T. Wu, Phys. Rev. **186** (1969), 1611.

9) E. Witten, Commun. Math. Phys. **117** (1988), 353; **118** (1988), 411; **121** (1989), 351; Phys. Lett. **206B** (1988), 601. For a review see, D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Reports **209C** (1991), 129.

10) D. Kabat and M. Ortiz, Nucl. Phys. **B 388** (1992), 570. See also R. Jackiw, D. Kabat and M. Ortiz, Phys. Lett. **B 277** (1992), 148.

11) D. Amati, M. Ciafaloni and G. Veneziano, Nucl. Phys. **B 347** (1990), 550; **B403** (1993), 707; Phys. Lett. **B289** (1992), 87;

12) M. Fabbrichesi, R. Pettorino, G. Veneziano and G.A. Vilkovisky, Nucl. Phys. **B419** (1994), 147.

13) S. Deser, Phys. Rev. Lett. **64** (1990), 611; S. Deser and J.G. McCarthy, Nucl. Phys. **B344** (1990), 747; M. Reuter, Phys. Rev. **D44** (1991), 1132.

14) J.M. Leinaas and J. Myrlheim, Nuovo Cimento **B37** (1977), 1; F. Wilczek, Phys. Rev. Lett. **48** (1982), 1144; **49** (1982), 957; F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983), 2250.

15) S. Forte, Rev. Mod. Phys. **64** (1992), 193; F. Wilceck, *Fractional Statistics and Anyon Superconductivity* (World Scientific Pub., 1990)

16) S. Deser, R. Jackiw and G. 'tHooft, Ann. Phys. **152** (1984), 220.

17) G. 'tHooft, Commun. Math. Phys. **117** (1988), 685; S. Deser and R. Jackiw, Commun. Math. Phys. **118** (1988), 495; D. Lancaster and N. Sasakura, Class. Quantum Grav. **8** (1991), 1481.

18) S. Deser, R. Jackiw and S. Templeton, Ann. Phys. **140** (1982), 372; Phys. Rev. Lett. **48** (1982), 975;

19) S. Deser and A.R. Steif, Class. Quantum Grav. **9** (1992), 153; S. Deser, J. McCarthy and A.R. Steif, Nucl. Phys. **B412** (1994), 305.

20) M. Zeni, Class. Quantum Grav. **10** (1993), 905.

Figure Captions

Fig. 1 Feynman diagrams summed up in the formula (17). The solid lines denote the scalar particles, the dashed lines the graviton exchange.

Fig. 2 The contour C of the σ-integration in Eq. (46) for the ultra-high energy scatterings